On computer algebra aided generation of exact solutions for Fredholm integro-differential equations

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Abstract. We introduce a topic in the intersection of symbolic mathematics and computation, in the fields of Boundary Value Problems for linear integral equations. Our computational approach gives emphasis to mathematical methodology and aims at both symbolic and graphical results. It is implemented in a widely used computer algebra system, *Mathematica*. We develop a solver for unique solutions of Fredholm integro-differential equations in a Mathematica notebook, that displays analytical formulations that can be called up directly. Our easy-to-use program provides in one entry, exact solutions for the abstract operator equation

$$Bu = Au - gF(Au) = f, D(B) = \{u \in D(A) : \Phi(u) = N\Psi(u)\}, u \in D(A), f \in Y \quad (1)$$

Our routine could make a research tool for a wide range of scientists, as BVP for integro-differential equations are often at the forefront of mechanics, physics, biosciences and finance. The code is written in *Mathematica* (v. 11.3). As the interpretation of the code is immediate, it allows ample space for improvements and customization.

Introduction

Applied sciences study phenomena mathematically formulated as Fredholm integro-differential equations subject to boundary conditions. In this study we use the symbolic computation program *Mathematica* [7] in order to generate the symbolic and graphical representation of the exact solutions of Fredholm integro-differential equations assuring their existence first. Our computational approach does not require built-in functions for symbolic solutions to ordinary differential equations and makes no assumptions for the initial conditions. The theoretical methodology comes from the work in [1, 2, 3, 4].

Analytical mathematical methods due to their complexity are not generally clear and immediately comprehensible to a large pool of scientists. In addition, computations work out faster with computer software, even faster when creating automatically the existence conditions and the intermediate results required before the analytic solution. The computer codes are fully presented and can be reproduced as they are in computational-based research practice. Results of solution steps obtained as outputs are created in a way as to be interpreted without the knowledge of the theoretical methodology. Mathematica has a dedicated function to symbolically solve an integro-differential equation (solves also Fredholm integral equations), DSolveValue (new feature in *Mathematica* v. 11). However, this function seems applicable only for low input parameters and does not give results in most models. Built-in functions NDSolve, NDSolveValue fail also. Numerical solutions of integro-differential equations (using numerous approximations e.g. Laplace transform methods), is a topic of interest in *Mathematica* fora (`https://mathematica.stackexchange.com/questions/24626`).

The paper is organized as follows: after a brief introduction into the mathematical context in section 1, we explain the workings of the code in section 2. Guidance on how to use and/or change the commands and adapt it to other cases is provided.

1. Mathematics Background

Consider the boundary value problem of the type

$$Bu = Au - gF(Au) = f, u \in D(A), f \in Y, \tag{2}$$

$$D(B) = \{u \in D(A) : \Phi(u) = N\Psi(u)\}, \tag{3}$$

$A : X \to X$ is an ordinary m order differential operator

$$Au(x) = \alpha_0 u^{(m)}(x) + \alpha_1 u^{(m-1)}(x) + ... + \alpha_m u(x), \quad \alpha_i \in \mathbb{R}, D(A) = X_A^m$$

where $X = C[a,b]$ or $X = L_p(a,b)$, $p \geq 1$, $X_A^m = C^m[a,b]$, if $X = C[a,b]$, or $X_A^m = W_p^m(a,b)$, if $X = L_p(a,b)$.

$F(Au) = col(F_1(Au), \ldots, F_n(Au))$ is a functional vector representing the integral part of the integro-differential equation, $g = (g_1, \ldots, g_n)$ is a vector of X^n, N a constant $m \times l$ matrix, $\Phi = col(\Phi_1, \ldots, \Phi_m)$, $\Psi = col(\Psi_1, \ldots, \Psi_l)$, are functional vectors with $\Psi(u)$ standing for the multipoint or integral part of the boundary conditions. Let $z = (z_1, z_2, \ldots, z_m)$ be a basis of $kerA$.

Boundary value problems $B : X \to X$ of the type of (1) for the specific case of $l = n$ have been studied by Vasiliev, Parasidis, Providas in [4], using the extension method. The extension method is a generalization of the direct method, which is presented in [4]. Here, problem (1) is investigated and solved also for the case $l \neq n$, $X \neq Y$. We assume multipoint or nonlocal integral boundary conditions, which allows us to consider a very large class of problems for the equation (1). The ultimate result is the exact solution of problem (1).

2. Program Explanation

In this section we propose how *Mathematica* resources can display analytic solutions of Fredholm IDEs that can be called up directly. All formulations come from simple code, with symbolic computations, matrix-vector multiplication, products of operators, as defined by the solution methods proposed in [4]. The subroutine solely uses simple *Mathematica* 's built-in functions as Inverse and Det.

The user must set the input parameters:

*The parameters in problem (*1*)*
m = the order m of the differential operator A,
l = the number of components or the dimension l of the functional vector Ψ,
n = the dimension n of the functional vector F,

*The structural elements of (*2*)*
F = functional vector F is the integral part of the IDE
g(.)=the functional vector on the left hand side of (2)
f(.)=the function on the right hand side of (2)
t= the list with the values of the variable in the boundary conditions

The structural elements of the boundary conditions
nmatrix = the $m \times l$ matrix N satisfying the matrix equation $\Phi(u) = N\Psi(u)$
Ψ(.) =the functional vector Ψ of the matrix equation $\Phi(u) = N\Psi(u)$

Par example, to define a particular F and Ψ with n=2, l=2 write
$F[function\_] := \{\int_0^1 x^2 * functiondx, \int_0^1 functiondx\}$
$\Psi[function\_] := \{function/.t \to ti[[1]], function/.t \to ti[[2]]\}$

The output consists of the following results:
1. W,V=the matrices in the condition for the injectivity of B (existence condition also)
2. Det[V], Det[W]=Determinants of W,V needed for the necessary and sufficient condition of injectivity of B
3. Automated testing for injectivity of B or the existence criterion as defined in [4]
4. solution= the analytic solution of Fredholm integro-differential equation
5. Plot the solution over the region of the initial conditions

The relevant output is created in a way as to be interpreted without the knowledge of the theoretical methodology. The criterion for injectivity of B that is tested and verified is the only requirement to apply Theorem 1 from [4] and formulate the unique solution.

The core part of the code is given in figure 1.

Conclusion

The computer codes proposed provide 1) formulation of the exact solution of Fredholm integro-differential equations with multipoint or nonlocal integral boundary conditions, 2) solvability exploration of infinite number of examples and 3) immediate construction of operators and functionals within the solution methodology.

```
In[∘]:= (*The solution method*)
In[∘]:= W := IdentityMatrix[n] - F[g[x]]
In[∘]:= V := IdentityMatrix[l] - Ψ[z].nmatrix
In[∘]:= z := Table[t^i / i!, {i, 0, m - 1}]
In[∘]:= inverseA[function_] := 1/(m - 1)! ∫_0^t (t - x)^(m-1) * function dx
In[∘]:= (*Testing necessary and sufficient conditions for operator Bu=Au-gF(Au) to be injective*)
In[∘]:= Det[W]
In[∘]:= Det[V]
In[∘]:= (*Testing the existence and uniqueness criterion*)
In[∘]:= If[Det[W] ≠ 0 && Det[V] ≠ 0, "The IDE has a unique solution", "The solution is not unique"]
In[∘]:= (*Here is the unique solution by the exact solution method*)
In[∘]:= solution := Simplify[
          inverseA[f[x]] + (inverseA[g[x]] + z.nmatrix.Inverse[V].Ψ[inverseA[g[x]]]).Inverse[W].F[f[x]] +
           z.nmatrix.Inverse[V].Ψ[inverseA[f[x]]]]
In[∘]:= Print["The exact solution of the IDE is" Flatten[solution]]
In[∘]:= Expand[solution[[1]]]
In[∘]:= Plot[%, {t, 0, 1}, AxesLabel → {"t", solution[[1]]}, PlotLabel → "u(t) over the region of the initial conditions"]
```

FIGURE 1. The exact solution routine in *Mathematica*

References

[1] I.N. Parasidis, P.C. Tsekrekos, Correct selfadjoint and positive extensions of non-densely defined symmetric operators. *Abstract and Applied Analysis*, Hindawi Publishing Corporation, no.7. p.p. 767-790, 2005.

[2] I.N. Parasidis, P.C. Tsekrekos, Correct and self-adjoint problems for quadratic operators. *Eurasian Mathematical Journal*, no.2. p.p. 122-135, 2010.

[3] I.N. Parasidis, E. Providas, Extension Operator Method for the Exact Solution of Integro-Differential Equations. In Pardalos P., Rassias T.(eds) *Contributions in Mathematics and Engineering: In Honor of Constantin Caratheodory*, Springer, Chambridge, p.p. 473-496, 2016.

[4] N.N. Vasiliev, I. N.Parasidis, E. Providas, Exact solution method for Fredholm integro-differential equations with multipoint and integral boundary conditions. Part 1. Extension method. *Informatsionno-upravliaiushchie sistemy [Information and Control Systems]*, no. 6, pp. 14–23, 2018.

[5] A.M. Wazwaz, Linear and nonlinear Integral Equations. *Methods and Applications*, Springer, Beijing, 2011.

[6] M. Benchohra, S.K. Ntouyas, Existence Results on the Semiinfinite Interval for First and Second Order Integrodifferential Equations in Banach spaces with nonlocal conditions. *Acta Universitatis Palackianae Olomucensis, Facultas Rerum Naturalium, Mathematica*, no.41, p.p. 13-19, 2002.

[7] Wolfram Research, Inc. Mathematica, Version 11.3, Champaign, Illinois, 2018.

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